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Abstract

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Introduction

We are interested to study $u(x,t)$, the evolution in time of the concentration, which is transported by diffusion and convection from a "sources site" made of a large number of similar "local sources". For this we consider a "local model" based on a general diffusion convection equation:

$$\partial_t u^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) + \operatorname{div}(b(x)u^\varepsilon) = f^\varepsilon; \tag{1}$$

$$u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^\varepsilon \cdot n(x) - b(x)\cdot n(x)u^\varepsilon + \lambda u^\varepsilon = 0. \tag{2}$$

where the sources density f^ε comes from a set of "local sources" periodically repeated and lying on a same plan Σ; $f^\varepsilon(x,t) = \bigcup_{\mathbf{j}\in\mathbb{Z}^2} f_{\mathbf{j}}(x,t)$. Assuming the release curve (source emission vs. space and time),$f_{\mathbf{j}}(,.)$, of each local source, being random, our aim is to give a mathematical model describing the global evolution of such a system .

In section 1, we define the geometry of the sources site and different randomness assumptions for the local sources. In this first section, we study a general "sources site" model assuming that the random field, $x \mapsto f^\varepsilon(x,t)$(or space trajectory), is statistically homogeneous and ergodic. In section 2, adding different mixing properties to these very general assumptions, we estimate the rate of convergence expectation for $(u^\varepsilon - u^0)$. In section 3,under the same assumptions, we prove, the convergence in distribution , for the corrector, $u^1 = (u^\varepsilon - u^0)/\varepsilon$, at any fixed point (x,t) to a centered Gaussian random variable . Finally, assuming the time trajectory, $t \mapsto f^\varepsilon(x,t)$is not random, we obtain the same convergence in distribution, for any finite joint distribution.

This modelisation is used for describing contaminants transport and migration in aquifers, see for instance [2], from a long-lived nuclear waste underground repository.In this case the above "local model" may be used for all the scaling up stages : from the set of containers to a repository unit, or from the set of repository units to a repository zone, or finally from the set of repository zones to the entire waste site. The general random behavior of the sources, as considered in sections 1 and 2, has only to be adapted to the different scaling up stages, by adding more precise assumptions on the randomness for each situation.

1 Definition of the problem

1.1 Description of the geometry

Given a smooth bounded domain $Q \subset \mathbb{R}^3$ with diameter $diam(Q) = R < \infty$, such that $Q^+ = \{x \in Q \, : \, x_3 > 0\}$ and $Q^- = \{x \in Q \, : \, x_3 < 0\}$ nonempty Lipschitz domains; now we describe the geometry of the sources supports inside this domain. First denote ε a small positive number (measuring the typical length of a source support or the adimensionalised period of the source supports); assuming that, in the local variables, each source has its support K_ε in a thin parallelepipedic set

$$K_\varepsilon = [0, s_1] \times [0, s_2] \times \varepsilon^{\gamma-1}[-s_3, s_3]$$

with $0 < s_{1,2} < 1$ and $\gamma \geq 1$; then by repeating periodically a single source support, we define a "sources site" support B_ε, and $\tilde{B}_\varepsilon$ a projection of the "sources site" on the middle plan $\Sigma = \{x \in Q \, : \, x_3 = 0\}$.
First we assume that, all the source supports K_ε of the entire "sources site" are situated inside a square $\Pi \subset \Sigma$, and not intersecting with the boundary $\partial\Pi$ of Π; then:

$$\widehat{B^j_\varepsilon} = \varepsilon([0, s_1] \times [0, s_2] + (j, 0)); \; \tilde{B}^j_\varepsilon = \widehat{B}^j_\varepsilon \bigcap \Pi = \varepsilon([0, s_1] \times [0, s_2] + (j, 0)) \bigcap \Pi, \quad (3)$$

$$\tilde{B}_\varepsilon = \bigcup_{j \in \mathbb{Z}^2} \tilde{B}^j_\varepsilon ; \tilde{B}_\varepsilon \bigcap \partial\Pi = \Phi; \quad (4)$$

$$B^j_\varepsilon = \tilde{B}^j_\varepsilon \times \varepsilon^\gamma[-s_3, s_3]; \; B_\varepsilon = \bigcup_{j \in \mathbb{Z}^2} B^j_\varepsilon . \quad (5)$$

$$S^j_\varepsilon = \varepsilon([0, 1]^2 + (j, 0)). \quad (6)$$

1.2 Description of the randomness

Starting with $(\Omega, \mathcal{F}, \mathbf{P})$ a standard probability space, for describing the randomness dependancy in space, assuming that the sources density f^ε is statistically homogeneous with respect to the 2D variable $x' = (x_1, x_2)$, we define on this probability space a discrete random ergodic dynamical system

$T_z; \mathbb{z} \in \mathbb{Z}^2$. Then by definition the random ergodic dynamical system T_z, is a collection of measurable maps $T_z : \Omega \to \Omega$ such that T_z
- preserves the measure $\mathbf{P}$ for all $z \in \mathbb{Z}^2$;
- has the group properties, $T_{z+y} = T_z \circ T_y$ for all z, y, $T_0 = \text{Id}$;
- is ergodic, i.e. the relation $\mathbf{P}(\mathcal{A})(1 - \mathbf{P}(\mathcal{A})) = 0$ holds for any invariant set $\mathcal{A} \in \mathcal{F}$.

1.3 Description of the equations

Under the above assumptions, we are now considering the initial-boundary problem, with a random right hand side:

$$\partial_t u^\varepsilon - \operatorname{div}(a(x)\nabla u^\varepsilon) + \operatorname{div}(b(x)u^\varepsilon) = f^\varepsilon, \qquad \text{in } Q \times (0, \infty); \tag{7}$$

$$u^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a} u^\varepsilon - b(x) \cdot n(x) u^\varepsilon + \lambda u^\varepsilon = 0 \quad \text{on } \partial Q \times (0, \infty), \tag{8}$$

where $a(x)$ (the diffusion tensor) is a uniformly and positive definite smooth matrix-function, $b(x)$ (the convection velocity) is a smooth vector field; and n_a and n are the external co-normal and normal respectively.

In the following, we will use the notation:

$$x' = (x_1, x_2) \in \Sigma; \; x = (x', x_3), \tag{9}$$

$$\text{and } \; \mathbf{z} = [x'/\varepsilon], \; \mathbf{x'} = [x']; \tag{10}$$

with $[.]$ denoting the integer part.
Regarding the random source term , as usual, we will not show explicitly the dependency in ω, the outcomes, and write the random variables as follow:

$$\phi^\varepsilon(x, t) = \phi(T_{\mathbf{z}}\omega, t), \; \Phi(x, t) = \phi(T_{\mathbf{x'}}\omega, t). \tag{11}$$

and

$$f^\varepsilon(x, t) = 1\!\!\mathrm{I}_{B_\varepsilon} \frac{1}{\varepsilon^\gamma} \phi(T_{\mathbf{z}}\omega, t), \; f(x, t) = 1\!\!\mathrm{I}_{\epsilon^{-1}B_\varepsilon} \frac{1}{\varepsilon^\gamma} \Phi(x, t). \tag{12}$$

We should notice first that the random function $f^\varepsilon(x, t)$, is independant of x_3. In the next section, corresponding to the actual problem, we will assume $x \mapsto f^\varepsilon(x,t)$(the space trajectory), to be statistically homogeneous and ergodic; but no special assumption will be done for the randomness of $t \mapsto \phi^\varepsilon(x,t)$. We will only assume the random function $\phi^\varepsilon(.,.)$to be uniformly bounded, i.e. that there exists nonrandom constants $\Lambda > 0$ and $C > 0$ such that, for any fixed $t \in (0,\infty)$ and any $(x,\omega) \in (Q \times \Omega)$,

$$|\phi^\varepsilon(x,t)| = |\phi(T_{\mathbf{z}}\omega,t)| \le Ce^{-\Lambda t}. \tag{13}$$

A classical result says that for each $\varepsilon > 0$ and each $\omega \in \Omega$ problem (7)-(8) has a unique solution $u^\varepsilon \in L^2(0,T;H^1(Q)) \cap C(0,T;L^2(Q))$.
Our first aim is now to show that the limit problem takes the form

$$\partial_t u^0 - \operatorname{div}(a(x)\nabla u^0) + \operatorname{div}(b(x)u^0) = F(t)\delta_\Sigma(x), \quad \text{in } Q \times (0,\infty); \tag{14}$$

$$u^0\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}u^0 - b \cdot nu^0 + \lambda u^0 = 0 \quad \text{on } \partial Q \times (0,\infty); \tag{15}$$

where $F(t)$ is the expectation (constant in space, due to homogeneity and ergodicity of the Random field Φ in space):

$$F(t) = s_1 s_2 \mathbf{E}\{\Phi(x,t)\},$$

and $\delta_\Sigma(x)$ is the surface Lebesgue measure with support $\Sigma \bigcap \Pi$.
More precisely, we are going to prove that u^ε converges to u^0, as $\varepsilon \to 0$, in $L^2(0,T;H^1(Q))$ norm. To this end we introduce an auxiliary 2D source term with support on $\Sigma \bigcap \Pi$

$$F^\varepsilon(x,t) = \mathbb{I}_{\tilde{B}_\varepsilon}\phi(T_{\mathbf{z}}\omega,t)\delta_\Sigma(x). \tag{16}$$

with $\tilde{B}_\varepsilon$ defined as as in (4),

Lemma 1. *Under assumption,(13) , for ay fixed* t*, the following bound holds:*

$$\|f^\varepsilon(\cdot,t) - F^\varepsilon(\cdot,t)\|_{H^{-1}(Q)} \le C\varepsilon^{\gamma/2}e^{-\Lambda t}$$

with C *a nonrandom constant.*

Proof. Letting

$$g^\varepsilon(x,t) = \begin{cases} +\displaystyle\int_{-1}^{x_3} f^\varepsilon(x',y,t)dy, & x_3 \le 0 \\ -\displaystyle\int_{x_3}^{-1} f^\varepsilon(x',y,t)dy, & x_3 > 0; \end{cases}$$

we have

$$\frac{\partial}{\partial x_3} g^\varepsilon(x,t) = f^\varepsilon(x,t) - F^\varepsilon(x,t), \qquad \text{and} |g^\varepsilon(x,t)| \le Ce^{-\Lambda t}.$$

It is also clear that $\mathrm{supp}(g^\varepsilon) \subset \Sigma$.
Therefore, for any $\varphi \in C_0^\infty(Q)$,

$$\Big| \int\limits_Q (f^\varepsilon(x,t) - F^\varepsilon(x,t))\varphi(x)dx \Big| = \Big| \int\limits_Q \frac{\partial}{\partial x_3} g^\varepsilon(x,t)\varphi(x)dx \Big| \le$$

$$\le \Big| \int\limits_Q g^\varepsilon(x,t)\frac{\partial}{\partial x_3}\varphi(x)dx \Big| \le \Big(\int\limits_\Sigma (g^\varepsilon(x,t))^2 dx \Big)^{1/2} \Big(\int\limits_Q |\nabla\varphi|^2 dx \Big)^{1/2} \le$$

$$\le C\varepsilon^{\gamma/2} \|\varphi\|_{H_0^1(Q)}\, e^{-\Lambda t},$$

which implies the statement. □

Moreover, by Birkhoff's theorem the function $(F^\varepsilon(x,t) - F(t)\delta_\Sigma(x))$ almost surely (a.s.) converges to 0 weakly in $L^2(\Sigma)$, as $\varepsilon \to 0$, for all $t > 0$. As a consequence we obtain the following statement.

Lemma 2. *Due to the assumption of ergodicity for the discrete random dynamical system T_z, and under assumption (13), we have the convergence, a.s., in L^2 norm:*

$$\lim_{\varepsilon\to 0} \|F^\varepsilon - F\delta_\Sigma\|_{L^2(0,T;H^{-1}(Q))} = 0. \tag{17}$$

Proof. For an arbitrary $\varphi \in L^2(0,T;H_0^1(Q))$ the relation holds

$$\Big| \int_0^T \int_Q (F^\varepsilon(x,t) - F(t)\delta_\Sigma(x))\varphi(x,t)\, dxdt \Big| =$$

$$= \Big| \int_0^T \int_\Sigma (F^\varepsilon(x',0,t) - F(t))\varphi(x',0,t)\, dx'dt \Big| \le$$

$$\le \int_0^T \|\varphi(\cdot,t)\|_{H^{1/2}(\Sigma)} \|(F^\varepsilon(\cdot,t) - F(t)\|_{H^{-1/2}(\Sigma)} dt \le$$

$$\le C\|\varphi\|_{L^2(0,T;H_0^1(Q))} \Big(\int_0^T \|(F^\varepsilon(\cdot,t) - F(t)\|^2_{H^{-1/2}(\Sigma)}\, dt \Big)^{1/2} \tag{18}$$

Due to the compactness of embedding of $L^2(\Sigma)$ into $H^{-1/2}(\Sigma)$, the a.s. weak convergence of $((F^\varepsilon(x', 0, t) - F(t))$ to 0 in $L^2(\Sigma)$ implies that a.s. for all $t \in [0, T]$
$$\lim_{\varepsilon\to 0} \|(F^\varepsilon(\cdot, t) - F(t))\|_{H^{-1/2}(\Sigma)} = 0.$$
Since $\|(F^\varepsilon(\cdot, t) - F(t))\|_{L^2(\Sigma)} \le Ce^{-\Lambda t}$, by the Lebesgue theorem we get a.s.
$$\lim_{\varepsilon\to 0} \int_0^T \|(F^\varepsilon(\cdot, t) - F(t))\|^2_{H^{-1/2}(\Sigma)} dt = 0$$
for any $T > 0$. The desired result now follows from (18). □

Remark 1. *Later on, we will also use the estimate*
$$\|F^\varepsilon(\cdot, t) - F(t)\delta_\Sigma\|_{H^{-1}(Q)} \le Ce^{-\Lambda t}, \quad \forall(\omega, t) \in \Omega \times (0, \rightarrow); \tag{19}$$
which is an easy consequence of (13).

We now proceed with the convergence result.

Theorem 1. *Under the same assumptions as in Lemma2*
$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u^0\|_{L^2(0,\infty;H^1(Q))} = 0 \quad a.s.,$$
with u^0 solution of (14),(15).

Proof. Subtracting (14) from (7) we conclude that the difference $(u^\varepsilon - u^0)$ satisfies the equation
$$\partial_t(u^\varepsilon - u^0) - \operatorname{div}(a(x)\nabla(u^\varepsilon - u^0)) - b(x)\nabla(u^\varepsilon - u^0) = f^\varepsilon - F(t)\delta_\Sigma(x), \quad \text{in } Q \times (0, \infty); \tag{20}$$
$$(u^\varepsilon - u^0)\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}(u^\varepsilon - u^0) + (\lambda - b \cdot n)(u^\varepsilon - u^0) = 0 \quad \text{on } \partial Q \times (0, \infty).$$
The standard energy estimate for this problem reads
$$\|u^\varepsilon - u^0\|_{L^2(0,T;H^1(Q))} \le C(T)\|f^\varepsilon - F\delta_\Sigma\|_{L^2(0,T;H^{-1}(Q))}; \tag{21}$$
it then follows from Lemma 1 and 2 that
$$\lim_{\varepsilon\to 0} \|u^\varepsilon - u^0\|_{L^2(0,T;H^1(Q))} = 0 \quad \text{a.s.} \tag{22}$$

for any $T > 0$. The proof of the convergence on the infinite time interval relies on the dissipative properties of the studied problem.
Consider a problem

$$\partial_t w - \operatorname{div}(a(x)\nabla w) - b(x)\nabla w = 0, \quad \text{in } Q \times (0,\infty); \tag{23}$$
$$w\big|_{t=0} = w_0, \qquad \frac{\partial}{\partial n_a} w + (\lambda - b\cdot n)w = 0 \quad \text{on } \partial Q \times (0,\infty).$$

Lemma 3. *A solution of problem (23) satisfies the estimate*

$$\|w(\cdot,t)\|_{C(Q)} \le Ce^{-\kappa t}\|w_0\|_{L^2(Q)}, \quad t \ge 1,$$

with constants $\kappa > 0$ and $C > 0$.

Proof of Lemma. Without loss of generality we may assume that $w_0 \ge 0$. Then, by the maximum principle, the solution w is positive for any time. Moreover, by the Harnack inequality and standard parabolic estimates,

$$\max_{x\in Q, 1\le t\le 2} w(x,t) \le C \min_{x\in Q, 1\le t\le 2} w(x,t) \le \|w_0\|_{L^2(Q)}$$

Integrating by parts the equation (23) over the set $\{(x,t) \,:\, x\in Q,\ 1\le t\le 2\}$ we obtain

$$\int_Q w(x,2)dt - \int_Q w(x,1)dt = -\lambda \int_1^2 dt \int_{\partial Q} w(x,t)d\sigma,$$

where $d\sigma$ is the surface volume element. It follows from the last two inequalities that

$$\int_Q w(x,2)dt \le \nu \int_Q w(x,1)dt$$

with $\nu < 1$. Iterating this procedure we conclude get

$$\int_Q w(x,N)dt \le \nu^N \int_Q w(x,1)dt.$$

Letting $-\kappa = \ln \nu$, the last estimate reads

$$\int_Q w(x,N)dt \le e^{-\kappa N} \int_Q w(x,1)dt.$$

¿From this estimate the statement of the Lemma easily follows by the Harnack inequality. $\square$

To complete the proof of Theorem denote by $G(x,y,t)$ the Green function of problem (23). By Lemma 3 and Harnack inequality we have

$$G(x,y,t) \le Ce^{-\kappa t}, \qquad t \ge 2,$$

and, by the parabolic estimates,

$$\|G(\cdot,y,t)\|_{H^1(Q)} \le Ce^{-\kappa t}, \qquad t \ge 2.$$

Combining this bound with (19) and the estimates of Lemma 1 and making use of the integral representation of a solution to parabolic problem (20) gives

$$\int_T^\infty \|u^\varepsilon(\cdot,t) - u^0(\cdot,t)\|^2_{H^1(Q)} dt \le Ce^{-\min(\Lambda,\kappa)t}.$$

Together with the convergence on finite intervals obtained in (22)this result implies the desired statement. □

2 Estimates for the rate of convergence

From the applications point of view the convergence result alone is not of great interest, if it is not accompanied by any rate of convergence estimates. In this section, under natural additional assumptions on the behaviour of the source term correlation function or mixing coefficients, see for instance [5] and [6] , we provide a number of bounds for the convergence rate.
Let us start by recalling the definition of the correlation function of a Random Field $(x',t) \in \mathbb{R}^2 \times \mathbb{R} \mapsto \tilde{f}(x',t)$

$$R(t,s,x',y') \equiv \mathbf{E}\big[(\tilde{f}(x',t) - \mathbf{E}\tilde{f}(x',t))(\tilde{f}(y',s) - \mathbf{E}\tilde{f}(y',s))\big]; \qquad (24)$$

which is $\equiv R(t,s,x'-y')$ if the Random Field $\tilde{f}$, is statistically homogeneous, in the variable x'.

2.1 Mixing assumptions

Here, and in the following, we will take :

$$\tilde{f}(x',t) \equiv \Phi(x,t) = \phi(T_{\mathbf{x}'}\omega,t); \qquad (25)$$

as defined in (9) and (11).
Moreover, we will use ,in the following, additional assumptions on the random field $t \mapsto \tilde{f}(x', t)$; i.e. :

- The upper bound for the correlation of $\tilde{f}(x', t)$ and $\tilde{f}(y', s)$ is only depending on the difference between x' and y' and does not depend on any particular choice of t or s;

and

- the closer the two times s, t are, the bigger is the dependancy (with global exponential decay); namely, for $\forall t, s \in [0, \rightarrow [$ and $\forall x', y' \in Q$,

$$\exists \bar{R}(y') \geq 0; \ |R(t, s, x', y')| \leq e^{-\Lambda \min(s,t)} \bar{R}(x' - y'). \tag{26}$$

In this last assumption,((26)) assumes that there is a locally, uniform in t and in s, estimate for the correlation of $\tilde{f}(\cdot, t)$ and $\tilde{f}(\cdot, s)$. We should also notice that the bound (26) is consistent with the previous assumption(13).
Concerning $\bar{R}(x', y')$ we will now assume that at least one of the following conditions holds true:

R0.

$$\bar{R}(y') = 0 \quad \text{if } |y'| > R_0.$$

for some $R_0 > 0$.

R1.

$$\int\limits_{\mathbb{R}^2} \bar{R}(y') dy' < \infty.$$

R2.

$$\bar{R}(y') \leq C(1 + |y'|)^{-\nu}, \qquad \nu > 0.$$

To obtain other estimates for the rate of convergence $u^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u^0$, including higher moments estimates for the discrepancy $|u^\varepsilon - u^0|$, we may assume that the functions $\tilde{f}(x', t) \equiv \Phi(x, t) = \phi(T_{\mathbf{x}'}\omega, t)$ possess one of the following mixing property:

M1. for each $t \geq 0$ the strong spatial mixing coefficient $\alpha_{t_0}(s)$ of $\tilde{f}(t, \cdot)$ decays fast enough so that

$$\alpha_{t_0}(s) \ < \ Ce^{-\Lambda t_0}(1 + s)^{-\nu_1}, \qquad \nu_1 > 0; \tag{27}$$

where the strong spatial mixing coefficient, $\alpha_{t_0}(s)$, is defined as follows:

$$\alpha_{t_0}(s) = \sup_{G_1,\, G_2} \sup_{\substack{\mathcal{E}_1 \in \mathcal{F}_{G_1} \\ \mathcal{E}_2 \in \mathcal{F}_{G_2}}} |\mathbf{P}(\mathcal{E}_1 \cap \mathcal{E}_2) - \mathbf{P}(\mathcal{E}_1)\mathbf{P}(\mathcal{E}_2)|$$

with $\mathcal{F}_{G_1} = \sigma\{\tilde{f}(y'_1, t_1) \,:\, y'_1 \in G_1,\, t_1 \geq t_0\}$, $\mathcal{F}_{G_2} = \sigma\{\tilde{f}(y'_2, t_2) \,:\, y'_2 \in G_2,\, t_2 \geq t_0\}$;and the first supremum taken over all sets $G_1,\, G_2 \subset \mathbb{R}^2$ such that $\mathrm{dist}(G_1, G_2) \geq s$.

M2. for each $t \geq 0$ the maximum spatial correlation coefficient $\beta_{t_0}(s)$ of $\tilde{f}(\cdot, t)$ decays fast enough so that

$$\beta_{t_0}(s) \,<\, Ce^{-\Lambda t_0}(1+s)^{-\nu_1}, \qquad \nu_1 > 0 \tag{28}$$

with the maximum spatial correlation coefficient is defined by:

$$\beta_{t_0}(s) = \sup_{G_1,\, G_2} \sup_{\xi,\, \eta} |\mathbf{E}(\xi\eta)|;$$

where the second supremum is taken over all random variables ξ and η which are respectively $\mathcal{F}_{G_1}$- and $\mathcal{F}_{G_2}$ -measurable and satisfy the conditions $\mathbf{E}\xi = \mathbf{E}\eta = 0$, $\|\xi\|_{L^\infty(\Omega)} = \|\eta\|_{L^\infty(\Omega)} = 1$; and the first supremum is taken over all sets $G_1,\, G_2 \subset \mathbb{R}^2$ such that $\mathrm{dist}(G_1, G_2) \geq s$.

Remark 2. *Notice that the condition* **R0** *is fulfilled if the strong mixing coefficient $\alpha_{t_0}(s)$ is equal to 0 for $s \geq R_0$, and also that the condition* **M1** *implies the condition* **R2** *with $\nu = \nu_1/3$.*

Replacing the random source density $f^\varepsilon(x, t)$, distributed in a small neighbourhood of the plane Σ, by a random source density $F^\varepsilon(x, t)$ concentrated on the plane Σ, as defined in (16), we consider the auxiliary problem:

$$\partial_t \hat{u}^\varepsilon - \mathrm{div}(a(x)\nabla \hat{u}^\varepsilon) + \mathrm{div}(b(x)\hat{u}^\varepsilon) = F^\varepsilon, \quad \text{in } Q \times (0, \infty); \tag{29}$$

$$\hat{u}^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}\hat{u}^\varepsilon - b(x)\cdot n(x)\hat{u}^\varepsilon + \lambda \hat{u}^\varepsilon = 0 \quad \text{on } \partial Q \times (0, \infty); \tag{30}$$

and obtain the next convergence estimates.

Lemma 4. *Under assumptions on the domain geometry in sec. 1.1 and assumptions on the coefficients of equations ((7)-((8) in sec. 1.3,with assumption(13), the bounds hold*

$$\|u^\varepsilon - \hat{u}^\varepsilon\|_{L^\infty(Q\times(0,T))} \le C\varepsilon^\gamma |\ln\varepsilon|, \|u^\varepsilon - \hat{u}^\varepsilon\|_{L^\infty(0,T;L^p(Q))} \le C(p)\varepsilon^\gamma, \quad 1 \le p < \infty. \tag{31}$$

Proof. Denote by $G(t-s,x,y)$ the Green function of problem (7)-(8). Using Aronson's estimates for fundamental solutions on finite time interval [1], the function $G(t,x,y)$ admits the following bounds

$$G(t,x,y) \le \frac{C}{t^{3/2}} \exp(-c\frac{|x-y|^2}{t})$$

with strictly positive C and c. Moreover,

$$|\nabla_y G(t,x,y)| \le \frac{C}{t^{3/2}} \frac{|x-y|}{t} \exp(-c\frac{|x-y|^2}{t}) \tag{32}$$

Clearly, the difference $(f^\varepsilon - F^\varepsilon)$ can be represented as follows

$$f^\varepsilon(x,t) - F^\varepsilon(x,t) = \frac{\partial}{\partial x_3}\vartheta\Big(\frac{x_3}{\varepsilon^\gamma}\Big)1\!\!1_{\tilde{B}^\varepsilon}(x')f(T_{\mathbf{z}}\omega,t),$$

with

$$\vartheta(r) = \begin{cases} r+1, & -1 \le r < 0, \\ r-1, & 0 \le r \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

By Green formula and estimates (13), (32), we have

$$|u^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)| = \Big| \int_0^t \int_Q G(t-s,x,y)\frac{\partial}{\partial y_3}\vartheta\Big(\frac{y_3}{\varepsilon^\gamma}\Big)1\!\!1_{\tilde{B}^\varepsilon}(y')f(T_{\mathbf{y'}/\varepsilon}\omega,s)\,dyds\Big|$$

$$= \Big| \int_0^t \int_Q \frac{\partial}{\partial y_3}G(t-s,x,y)\vartheta\Big(\frac{y_3}{\varepsilon^\gamma}\Big)1\!\!1_{\tilde{B}^\varepsilon}(y')f(T_{\mathbf{y'}/\varepsilon}\omega,s)\,dyds\Big| \le$$

$$\le \int_0^t \int_Q \frac{C}{(t-s)^2}\frac{|x-y|}{\sqrt{t-s}} \exp\Big(-c\frac{|x-y|^2}{t-s}\Big)1\!\!1_{\{|y_3|\le\varepsilon^\gamma\}} \exp(-\Lambda s)\,dsdy.$$

Integrating first in time and then in space, after straightforward computation, one gets

$$\int_0^t \frac{1}{(t-s)^2}\frac{|x-y|}{\sqrt{t-s}}\exp\Big(-c\frac{|x-y|^2}{t-s}\Big)\exp(-\Lambda s)\,ds \le$$

$$\int_0^t \frac{1}{s^2}\frac{|x-y|}{\sqrt{s}}\exp\Big(-c\frac{|x-y|^2}{s}\Big)\,ds \le \frac{1}{|x-y|^2}\int_0^\infty \frac{1}{s^{5/2}}\exp\Big(-\frac{c}{s}\Big)\,ds = \frac{c_2}{|x-y|^2}.$$

For $|x_3| \le 2\varepsilon^\gamma$ this gives (31)

$$|u^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)| \le C\int_0^{2\varepsilon^\gamma} dy_3 \int_{|y'|\le R} \frac{1}{|y|^2}\,dy' \le C(Q,\gamma)\varepsilon^\gamma|\ln(\varepsilon)|, \tag{33}$$

where, here and later on, $R = \mathrm{diam}(Q)$.
For $|x_3| \ge 2\varepsilon^\gamma$ we have

$$|u^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)| \le C\varepsilon^\gamma \int_{|y'|\le R} \frac{1}{x_3^2 + |y'|^2}\,dy' \le C(Q)\varepsilon^\gamma|\ln(|x_3|)|. \tag{34}$$

Finally, (33)–(34) yield

$$|u^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)| \le C(Q,\gamma)\varepsilon^\gamma|\ln(\max\{|x_3|, 2\varepsilon^\gamma\})|; \tag{35}$$

and in particular,

$$\|u^\varepsilon - \hat{u}^\varepsilon\|_{L^\infty(0,\infty;L^p(Q))} \le C(p)\varepsilon^\gamma \tag{36}$$

for all $p \in [1,\infty)$; which is (31) and completes the proof. □

Remark 3. *The proof of Lemma 4has nothing to do with the randomness of f^ε. We only used the Green function properties of the initial problem, the structure of the support of f^ε, and the fact that this function is bounded.*

Next we denote

$$\check{F}^\varepsilon(t) = \mathbf{E}\{\Phi(x,t)\}\mathbb{1}_{\tilde{B}_\varepsilon}(x')\delta_\Sigma(x) \tag{37}$$

and consider a deterministic auxiliary problem withF^ε as non random source term:

$$\partial_t \check{u}^\varepsilon - \operatorname{div}(a(x)\nabla \check{u}^\varepsilon) + \operatorname{div}(b(x)\check{u}^\varepsilon) = \check{F}^\varepsilon, \quad \text{in } Q\times(0,\infty); \tag{38}$$

$$\check{u}^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}\check{u}^\varepsilon - b(x)\cdot n(x)\check{u}^\varepsilon + \lambda\check{u}^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{39}$$

We will now use the following statement derived from [7] and [8].

Proposition 1. *Let be $\check{u}^\varepsilon$ solution of (38)-(39) and u^0 solution of ((7)-((8); then*

$$\|\check{u}^\varepsilon - u^0\|_{L^\infty(0,T;L^2(Q))} \le C\varepsilon. \tag{40}$$

In view of Lemma 4 and proposition 1, in order to estimate the discrepancy $\|u^\varepsilon - u^0\|_{L^\infty(0,T;L^2(Q))}$, it suffices to obtain an upper bound for the expression $\|\hat{u}^\varepsilon - \check{u}^\varepsilon\|_{L^\infty(0,T;L^2(Q))}$. This is the main and most technical part of this section.

Proposition 2. *Let $\hat{u}^\varepsilon$ solution of (29)-(30) and $\check{u}^\varepsilon$ solution of (38)-(39), with condition* **R0** *fulfilled; then the following estimate holds*

$$\mathbf{E}\big\{\|\hat{u}^\varepsilon - \check{u}^\varepsilon\|^2_{L^2(0,T;L^2(Q))}\big\} \le C\varepsilon^2. \tag{41}$$

Proof. The difference $U^\varepsilon \equiv (\check{u}^\varepsilon - \hat{u}^\varepsilon)$ solves the problem

$$\partial_t U^\varepsilon - \operatorname{div}(a(x)\nabla U^\varepsilon) + \operatorname{div}(b(x)U^\varepsilon) = F^\varepsilon - \check{F}^\varepsilon, \quad \text{in } Q\times(0,\infty); \tag{42}$$

$$U^\varepsilon\big|_{t=0} = 0, \qquad \frac{\partial}{\partial n_a}U^\varepsilon - b(x)\cdot n(x)U^\varepsilon + \lambda U^\varepsilon = 0 \quad \text{on } \partial Q\times(0,\infty). \tag{43}$$

Our aim is to estimate the expression $\mathbf{E}\big\{\|U^2(t,\cdot)\|^2_{L^2(Q)}\big\}$. To this end we first obtain a point-wise in x bound. Using the notation $F_0^\varepsilon = F^\varepsilon - \check{F}^\varepsilon$, by the Green formula we have

$$\mathbf{E}\big\{U^2(x,t)\big\} = \mathbf{E}\bigg\{\bigg(\int\limits_0^t\int\limits_Q G(t-s,x,y)F_0^\varepsilon(s,y)dyds\bigg)^2\bigg\} \le$$

$$\le \mathbf{E}\bigg\{\int\limits_0^t\int\limits_Q\int\limits_0^t\int\limits_Q G(t-s,x,y)G(t-r,x,z)F_0^\varepsilon(s,y)F_0^\varepsilon(r,z)\,dydsdzdr\bigg\}$$

$$\leq C\int_0^t\int_{Q'}\int_0^t\int_{Q'} G(t-s,x,(y',0))G(t-r,x,(z',0))\bar{R}\Big(\frac{|z'-y'|}{\varepsilon}\Big)\,dy'dsdz'dr$$

$$= C\int_0^t\int_{Q'}\int_0^t\int_{Q'} G(s,x,(y',0))G(r,x,(z',0))\bar{R}\Big(\frac{|z'-y'|}{\varepsilon}\Big)\,dy'dsdz'dr$$

$$\leq C_1\int_0^t\int_{Q'}\int_0^t\int_{Q'} \frac{1}{s^{3/2}r^{3/2}}\exp\Big(-c\frac{x_3^2+|x'-y'|^2}{s}\Big)\times$$

$$\exp\Big(-c\frac{x_3^2+|x'-z'|^2}{r}\Big)\bar{R}\Big(\frac{|z'-y'|}{\varepsilon}\Big)\,dy'dsdz'dr;$$

here we have denoted $Q'=Q\cap\{y_3=0\}$. We first integrate in s. This gives

$$\int_0^t \frac{1}{s^{3/2}}\exp\Big(-c\frac{x_3^2+|x'-y'|^2}{s}\Big)\,ds=$$

$$\frac{1}{(x_3^2+|x'-y'|^2)^{1/2}}\int_0^{t/(x_3^2+|x'-y'|^2)} \frac{1}{s^{3/2}}\exp\Big(-\frac{c}{s}\Big)\,ds\leq \frac{C_2}{(x_3^2+|x'-y'|^2)^{1/2}}$$

with $C_2=\int_0^\infty t^{-3/2}\exp(-c/s)ds$. Substituting this estimate in the previous inequality, one gets

$$\mathbf{E}\big\{U^2(x,t)\big\}\leq C\int_{Q'}\int_{Q'}\frac{\bar{R}\Big(\frac{|z'-y'|}{\varepsilon}\Big)dy'}{(x_3^2+|x'-y'|^2)^{1/2}}\frac{dz'}{(x_3^2+|x'-z'|^2)^{1/2}} \tag{44}$$

Without loss of generality we assume that $0\in Q$. If we denote $Q_0=\{y'\in\mathbb{R}^2,|y'|<2\mathrm{diam}(Q)\}$ and perform the change the variables $\tilde{y}'=y'-x'$, $\tilde{z}'=z'-x'$, then

$$\mathbf{E}\big\{U^2(x,t)\big\}\leq C\int_{Q_0}\int_{Q_0}\frac{1}{(x_3^2+|\tilde{y}'|^2)^{1/2}}\frac{1}{(x_3^2+|\tilde{z}'|^2)^{1/2}}\bar{R}\Big(\frac{|\tilde{z}'-\tilde{y}'|}{\varepsilon}\Big)\,d\tilde{y}'d\tilde{z}' \tag{45}$$

For brevity denote $Q_0^{2,\varepsilon} = \{(\tilde{y}', \tilde{z}') \in Q_0 \times Q_0 \,:\, |\tilde{y}' - \tilde{z}'| \le R_0\varepsilon\}$. Due to the assumption **R0**, we have

$$\mathbf{E}\big\{U^2(x,t)\big\} \le C \int\limits_{Q_0^{2,\varepsilon}} \frac{1}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{1}{(x_3^2 + |\tilde{z}'|^2)^{1/2}}\, d\tilde{y}' d\tilde{z}'.$$

In order to achieve an upper bound for this integral we divide the integration area into two parts, namely $Q_1^{2,\varepsilon} = Q_0^{2,\varepsilon} \cap \{|\tilde{y}'| < 2R_0\varepsilon,\, |\tilde{z}'| < 2R_0\varepsilon\}$ and $Q_2^{2,\varepsilon} = Q_0^{2,\varepsilon} \setminus Q_1^{2,\varepsilon}$. The integral over $Q_1^{2,\varepsilon}$ can be estimated as follows

$$\int\limits_{Q_1^{2,\varepsilon}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{d\tilde{z}'}{(x_3^2 + |\tilde{z}'|^2)^{1/2}} \le \Bigg(\int\limits_{\{|y'| < 2R_0\varepsilon\}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \Bigg)^2 =$$

$$= \Bigg(\int\limits_0^{2R_0\varepsilon} \frac{r\,dr}{(x_3^2 + r^2)^{1/2}} \Bigg)^2 = \Bigg(\int\limits_0^{4R_0^2\varepsilon^2} \frac{ds}{2(x_3^2 + s)^{1/2}} \Bigg)^2 \le C(R_0)\varepsilon^2; \qquad (46)$$

the explicit formula for the last integral has also been used here. For any $(x', y') \in Q_2^{2,\varepsilon}$ it holds

$$\frac{1}{(x_3^2 + |y'|^2)^{1/2}} \frac{dy'}{(x_3^2 + |z'|^2)^{1/2}} \le C(R_0) \frac{1}{(x_3^2 + |y'|^2)}$$

Hence,

$$\int\limits_{Q_2^{2,\varepsilon}} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)^{1/2}} \frac{d\tilde{z}'}{(x_3^2 + |\tilde{z}'|^2)^{1/2}} \le \int\limits_{Q_2^{2,\varepsilon}} \frac{d\tilde{y}' d\tilde{z}'}{(x_3^2 + |\tilde{y}'|^2)} \le$$

$$\le C(R_0)\varepsilon^2 \int\limits_{|\tilde{y}'| \ge 2R_0\varepsilon} \frac{d\tilde{y}'}{(x_3^2 + |\tilde{y}'|^2)} = C(R_0)\varepsilon^2 \int\limits_{2R_0\varepsilon}^{2\mathrm{diam}(Q)} \frac{r\,dr}{(x_3^2 + r^2)} \le \qquad (47)$$

$$\le C(R_0, Q)\varepsilon^2 \ln(x_3^2 + \varepsilon^2).$$

Combining (47) and (46), we arrive at the desired point-wise upper bound:

$$\mathbf{E}\big\{U^2(x,t)\big\} \le C(R_0, Q)\varepsilon^2 \ln(x_3^2 + \varepsilon^2). \qquad (48)$$

Now, the estimate (41) is straightforward, we should just integrate (48) over Q. □

The above statements allow us to estimate the rate of convergence in Theorem 1.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled, and assume in addition that the condition* **R0** *holds true. Then*

$$\mathbf{E}\Big\{\|u^\varepsilon-u^0\|^2_{L^2(0,T;L^2(Q))}\Big\}\leq C(R_0,Q)(\varepsilon^2+\varepsilon^{2\gamma}),$$

where u^ε is a solution of the original problem (7)-(8), and u^0 is a solution of homogenized problem (14)-(15).

Proof. This assertion is a straightforward consequence of Lemma 4 and Propositions 1 and 2. □

Taking into account the dissipative properties of the boundary conditions (8) and the bounds (13), (26), and following the line of the proof of Proposition 2, one can obtain the estimate

Theorem 3. *Under the assumptions of Theorem 2 the inequality*

$$\mathbf{E}\Big\{\|u^\varepsilon-u^0\|^2_{L^2((t,\infty);L^2(Q))}\Big\}\leq C(R_0,Q)\exp(-\kappa t)(\varepsilon^2+\varepsilon^{2\gamma}),\quad \kappa>0. \tag{49}$$

holds with $\kappa>0$ which only depends on Λ, the operator in (7)-(8) and the domain Q.

Our next goal is to relax the mixing assumptions on the source function $f(x,\omega)$. We want to show that the statement of the last theorem remains valid if the correlation function of f or its strong mixing coefficient satisfy certain polynomial decay conditions.

Theorem 4. *Suppose that either the condition* **R2** *is fulfilled with $\nu>2$ or the strong spatial mixing coefficient $\alpha_{t_0}(s)$ satisfies the upper bound*

$$\alpha_{t_0}(s)\leq C(1+s)^{-\nu_1}$$

with $\nu_1>6$. Then the inequality holds

$$\mathbf{E}\Big\{\|u^\varepsilon-u^0\|^2_{L^2(0,T;L^2(Q))}\Big\}\leq C(\nu,Q)(\varepsilon^2+\varepsilon^{2\gamma}),$$

Proof. It suffices to show that the estimate (48) holds. To this end we consider the auxiliary problem (42)-(43) with in the R.H.S.$F^\varepsilon - \check{F}^\varepsilon$ and notice that the upper bounds (44) and (45) are valid with assumptions of theorem 4. It follows from (45) and the standing assumptions that

$$\mathbf{E}\{U^2(x,t)\} \le C \int\limits_{Q_0}\int\limits_{Q_0} \frac{1}{(x_3^2+|\tilde{y}|^2)^{1/2}} \frac{1}{(x_3^2+|\tilde{z}|^2)^{1/2}} \Big(\frac{1}{(1+\varepsilon^{-1}|\tilde{z}-\tilde{y}|)^\nu}\Big)\, d\tilde{y}d\tilde{z} =$$

$$= C\varepsilon^2 \int\limits_{\varepsilon^{-1}Q_0}\int\limits_{\varepsilon^{-1}Q_0} \frac{1}{(X_3^2+|\tilde{y}|^2)^{1/2}} \frac{1}{(X_3^2+|\tilde{z}|^2)^{1/2}} \Big(\frac{1}{(1+|\tilde{z}-\tilde{y}|)^\nu}\Big)\, d\tilde{y}d\tilde{z}$$

with $\nu > 2$ and $X_3 = x_3/\varepsilon$; for the notation simplicity we write $\tilde{y}$ and $\tilde{z}$ instead of $\tilde{y}'$ and $\tilde{z}'$. We divide the domain $\varepsilon^{-1}Q_0 \times \varepsilon^{-1}Q_0$ into three parts, namely

$$\varepsilon^{-1}Q_0 \times \varepsilon^{-1}Q_0 = Q_1 \cup Q_2 \cup Q_3 =$$

$$\Big\{(\tilde{y},\tilde{z}) \,:\, |\tilde{y}| \le \frac{1}{2}|\tilde{z}|\Big\} \cup \Big\{(\tilde{y},\tilde{z}) \,:\, \frac{1}{2}|\tilde{z}| \le |\tilde{y}| \le 2|\tilde{z}|\Big\} \cup \Big\{(\tilde{y},\tilde{z}) \,:\, |\tilde{y}| \ge 2|\tilde{z}|\Big\},$$

and estimate the contribution of each subdomain separately. In Q_1 we have $|\tilde{z}' - \tilde{y}'| \ge \frac{1}{2}|\tilde{z}'|$, thus

$$\int\limits_{\varepsilon^{-1}Q_0} \frac{C\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)^{1/2}} \int\limits_{\{2|\tilde{y}|\le|\tilde{z}|\}} \frac{1}{(X_3^2+|\tilde{y}|^2)^{1/2}} \frac{1}{(1+|\tilde{z}-\tilde{y}|)^\nu}\, d\tilde{y} \le$$

$$\le \int\limits_{\varepsilon^{-1}Q_0} \frac{C\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)^{1/2}} \int\limits_{\{2|\tilde{y}|\le|\tilde{z}|\}} \frac{1}{(X_3^2+|\tilde{y}|^2)^{1/2}} \frac{C(\nu)}{(1+|\tilde{z}|)^\nu}\, d\tilde{y} \le$$

$$\le \int\limits_{\varepsilon^{-1}Q_0} \frac{C(\nu)\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)^{1/2}(1+|\tilde{z}|)^\nu} \int\limits_0^{|\tilde{z}|/2} \frac{r dr}{(X_3^2+r^2)^{1/2}} \le$$

$$\int\limits_{\varepsilon^{-1}Q_0} \frac{C(\nu)\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)^{1/2}(1+|\tilde{z}|)^\nu} C_1(\nu)(X_3^2+|\tilde{z}|^2)^{1/2} \le C_2(\nu)\varepsilon^2.$$

The contribution of Q_3 can be estimated in the same way if we exchange the order of integration in the variables $\tilde{y}$ and $\tilde{z}$. It remains to estimate the integral over Q_2. We have

$$\int\limits_{Q_2} \frac{C\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)^{1/2}} \frac{d\tilde{y}}{(X_3^2+|\tilde{y}|^2)^{1/2}} \frac{1}{(1+|\tilde{z}-\tilde{y}|)^\nu} \le$$

$$\leq \int\limits_{Q_2} \frac{4C\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)} \frac{d\tilde{y}}{(1+|\tilde{z}-\tilde{y}|)^\nu} \leq \int\limits_{\varepsilon^{-1}Q_0} \frac{4C\varepsilon^2 d\tilde{z}}{(X_3^2+|\tilde{z}|^2)} \int\limits_{2\varepsilon^{-1}Q_0} \frac{d\hat{y}}{(1+|\hat{y}|)^\nu} \leq$$

$$\leq C(\nu)\varepsilon^2 \int\limits_{\varepsilon^{-1}Q_0} \frac{d\tilde{z}}{(X_3^2+|\tilde{z}|^2)} \leq C_3(\nu)(1+|\ln(|x_3|)|).$$

Combining the above estimates we conclude that

$$\mathbf{E}\big\{U^2(x,t)\big\} \leq C(\nu)\varepsilon^2\big(1+|\ln(|x_3|)|\big).$$

This yields the desired statement if the assumption **R2** is fulfilled with $\nu > 2$. To complete the proof, we use the fact that the condition **M1** implies the condition **R2** with $\nu = \nu_1/3$. $\square$

Theorem 5. *Assume that at least one of the conditions* **R1**, **R2** *with* $\nu > 2$, *or* **M1** *with* $\nu_1 > 6$ *is satisfied. Then the discrepancy* $(u^\varepsilon - u^0)$ *admits the estimate*

$$\mathbf{E}\Big\{\|u^\varepsilon - u^0\|^2_{L^2((t,\infty);L^2(Q))}\Big\} \leq C\exp(-\kappa t)(\varepsilon^2+\varepsilon^{2\gamma}), \quad \kappa > 0.$$

By following the line of the proof of Theorem 4 one can show that under the condition **R1** the upper bound holds

$$\mathbf{E}\Big\{\|u^\varepsilon - u^0\|^2_{L^2(0,T;L^2(Q))}\Big\} \leq C(\varepsilon^2+\varepsilon^{2\gamma});$$

and finally, combining the above local in time estimates and the statement of Lemma 3, and considering the presence of exponentially decaying factors in (13), (26) and (27), we arrive at the result.

In order to improve the statements of Lemma 4 and Proposition 1, we will need an additional assumption. Let's denote $Q_\delta = \{x \in Q\,:\,|x_3| > \delta\}$ and $Q_\delta^- = \{x \in Q\,:\,|x_3| \leq \delta\}$; we will assume then:

H1. there exists $\delta_0 > 0$ such that $Q_{\delta_0}^- = \Sigma \times [-\delta_0,\delta_0]$;i.e. in the vicinity of the hyperplane Σ the domain Q has a cylindrical shape.

Lemma 5. *Under the assumptions of Lemma 4 and the additional assumption* **H1**, *for any* $\delta > 0$, *and almost surely, the two relations hold:*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\gamma}\|u^\varepsilon - \hat{u}^\varepsilon\|_{L^\infty(Q_\delta)} = 0. \tag{50}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^\gamma}\|u^\varepsilon - \hat{u}^\varepsilon\|_{L^p(Q)} = 0, \qquad p \in [1,+\infty). \tag{51}$$

Proof. In view of (35) the second relation is a consequence of the first one. To prove (50) we make use of the representation

$$f^\varepsilon(x,t) - F^\varepsilon(x,t) = \varepsilon^\gamma \frac{\partial^2}{\partial x_3^2}\vartheta_1\Big(\frac{x_3}{\varepsilon^\gamma}\Big)\mathbb{I}_{\tilde B^\varepsilon}(x')f(T_{\mathbf{x}'/\varepsilon}\omega,t),$$

with

$$\vartheta_1(r) = \begin{cases} \frac{1}{2}(r+1)^2, & -1 \le r < 0,\\ \frac{1}{2}(r-1)^2, & 0 \le r \le 1,\\ 0, & \text{otherwise}.\end{cases}$$

By the Green formula and due to the assumption **H1**, for sufficiently small $\varepsilon > 0$ we have

$$|u^\varepsilon(x,t) - \hat u^\varepsilon(x,t)| = \varepsilon^\gamma \Big|\int_0^t\int_Q G(t-s,x,y)\frac{\partial^2}{\partial y_3^2}\vartheta_1\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{I}_{\tilde B^\varepsilon}(y')f(T_{\mathbf{y}'/\varepsilon}\omega,s)\,dyds\Big|$$

$$= \varepsilon^\gamma \Big|\int_0^t\int_Q \frac{\partial^2}{\partial y_3^2}G(t-s,x,y)\vartheta_1\Big(\frac{y_3}{\varepsilon^\gamma}\Big)\mathbb{I}_{\tilde B^\varepsilon}(y')f(T_{\mathbf{y}'/\varepsilon}\omega,s)\,dyds\Big|.$$

Since the Green function satisfies the estimate

$$\Big|\frac{\partial^2}{\partial y_3^2}G(t,x,y)\Big| \le C(\delta)$$

for all x, y and t such that $|x-y| \ge \delta$, $t > 0$, then

$$|u^\varepsilon(x,t) - \hat u^\varepsilon(x,t)| \le C(\delta_0/2)\varepsilon^{2\gamma}$$

for all $x \in Q_{\delta_0}$, $t > 0$ and $\varepsilon < \delta_0/2$. This implies the desired statement. □

Now we proceed by studying the asymptotic behaviour of the normalized difference $\varepsilon^{-1}(\breve u^\varepsilon - \breve u^\varepsilon)$.
Denote

$$\bar{\bar c}(t,s) = \lim_{N\to\infty}\frac{1}{N^2}\int_{[0,N]^4} R(t,s,y-z)\mathbb{I}_{\tilde B}(y)\mathbb{I}_{\tilde B}(z)dy_1dy_2dz_1dz_2;$$

it is easy to verify that under condition **R1** the above limit exists and admits the upper bound

$$\bar{\bar c}(t,s) \le 4\int_0^\infty\int_0^\infty \bar R(y)dy_1dy_2.$$

3 Corrector's limit law

Theorem 6. *Assume that one of the conditions* **M1** *or* **M2** *is fulfilled with* $\nu_1 > 2$. *Then for each* $t > 0$ *and* $x \in Q$, $x_3 \neq 0$, *the normalized difference* $\varepsilon^{-1}(\check{u}^\varepsilon - \hat{u}^\varepsilon)$, *with* $\hat{u}^\varepsilon$ *solution of (29)-(30) and* $\check{u}^\varepsilon$ *solution of (38)-(39), converges in law towards a Gaussian random variable with zero mean and covariance*

$$\sigma^2(t,x) = \int_0^t \int_0^t \int_\Sigma G(t-s,x,y')G(t-r,x,y')\bar{\bar{c}}(s,r)dy'dsdr.$$

Proof. The proof is obtained by adapting the lines of the Central Limit Theorem, with $\varepsilon^{-1} \approx N; N^2 \approx$ numbers of local sources, for stationnary homogeneous random field in [[3]] to the random variable

$$X_{\mathbf{j}}(x,t) = \int_0^t \int_\Sigma G(t-s,x,y') \times \phi(T_{\mathbf{j}}\omega,t) 1\!\!1_{\tilde{B}_\varepsilon^{\mathbf{j}}}(y')dy'dt; \ \mathbf{j} \in \mathbb{Z}^2. \tag{52}$$

We only have to find the limit

$$\lim_{\varepsilon\to 0} \mathbf{E}\big\{(\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t))^2\big\}.$$

It is straightforward to compute

$$\mathbf{E}\big\{(\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t))^2\big\} = \frac{1}{\varepsilon^2}\int_\Sigma\int_\Sigma\int_0^t\int_0^t G(t-s,x,y')G(t-r,x,z')\times$$

$$R\Big(s,r,\frac{y'-z'}{\varepsilon}\Big) 1\!\!1_{\tilde{B}_\varepsilon}(z') 1\!\!1_{\tilde{B}_\varepsilon}(y')dz'dy'dsdr$$

Since $\nu_1 > 2$, then the assumption **R1** is satisfied. Also, for $x_3 \neq 0$, the function $G(t-s,x,y')$ is continuously differentiable in s and y'. Therefore,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_\Sigma\int_\Sigma G(t-s,x,y')G(t-r,x,z')R\Big(s,r,\frac{y'-z'}{\varepsilon}\Big) 1\!\!1_{\tilde{B}_\varepsilon}(z') 1\!\!1_{\tilde{B}_\varepsilon}(y')dz'dy' =$$

$$= \int_\Sigma G(t-s,x,y')G(t-r,x,y')\bar{\bar{c}}(s,r)dy'$$

for all s and r, $0 \leq s, r \leq t$. The relation

$$\lim_{\varepsilon \to 0} \mathbf{E}\big\{(\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \check{u}^\varepsilon(x,t))^2\big\} = \sigma^2(t,x)$$

now follows from the Lebesgue theorem. □

Unfortunately, we cannot claim that under conditions of the last theorem any finite dimensional distributions of $\varepsilon^{-1}(\check{u}^\varepsilon - \hat{u}^\varepsilon)$ converge in law to Gaussian vectors. This is due to the randomness dependence of temporal variable which is not specified. However, in some particular cases the limit field is Gaussian. For instance, if in (7),

$$f^\varepsilon(x,t) = \mathbb{I}_{B_\varepsilon} \frac{1}{\varepsilon^\gamma} \lambda(T_{\mathbf{x}'/\varepsilon}\omega)\Psi(t), \tag{53}$$

with any $\Psi(t)$ being a deterministic function; then the function $(\varepsilon^{-1}(\check{u}^\varepsilon(x,t) - \hat{u}^\varepsilon(x,t)))$ will converge in law to a Gaussian field.

Proposition 3. *Let $f^\varepsilon(x,t)$, in (7), be of the form (53). Then for any finite set $(x^1,t^1),(x^2,t^2),\ldots,(x^N,t^N)$ the random vector $\big\{(\varepsilon^{-1}(\hat{u}^\varepsilon(x^j,t^j) - \check{u}^\varepsilon(x^j,t^j)))\big\}$, $j = 1,\ldots,N$, , with $\hat{u}^\varepsilon$ solution of (29)-(30) and $\check{u}^\varepsilon$ solution of (38)-(39), converges in law towards a Gaussian vector with the covariance matrix $\{\sigma^{ij}\}$,*

$$\sigma^{ij} = \int_0^{t^i}\int_0^{t^j}\int_\Sigma G(t^i - s, x^i, y')G(t^j - r, x^j, y')\bar{\bar{c}}(s,r)dy'drds.$$

Proof. The proof is the same as that of Theorem 6. □

Proposition 3 statement will remain valid for a R.H.S., in (7), of the form

$$f^\varepsilon(x,t) = \mathbb{I}_{B_\varepsilon} \frac{1}{\varepsilon^\gamma} \sum_{i=1}^K \lambda_i(T_{\mathbf{x}'/\varepsilon}\omega)\Psi_i(t),$$

with with all functions $\Psi_i(t)$ being deterministic . We proceed with estimating the difference $(u^0 - \check{u}^\varepsilon)$. Making use of the standard ergodic and homogenization as in [7] or [4], one can prove the following assertion.

Lemma 6. *For any $t > 0$ and $x \in (Q \setminus \Sigma)$ the relation holds*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |\check{u}^\varepsilon(x,t) - u^0(x,t)| = 0.$$

Theorem 7. *Let $f^\varepsilon(x,t)$, in (7), be of the form (53). Then for any finite set $(x^1, t^1), (x^2, t^2), \ldots, (x^N, t^N)$ the random vector $\big\{(\varepsilon^{-1}(\check{u}u^\varepsilon(x^j, t^j) - u^0)\big\}$, $j = 1, \ldots, N$, converges in law towards a Gaussian vector with the covariance matrix $\{\sigma^{ij}\}$,*

$$\sigma^{ij} = \int_0^{t^i} \int_0^{t^j} \int_\Sigma G(t^i - s, x^i, y') G(t^j - r, x^j, y') \bar{\bar{c}}(s,r) dy' dr ds$$

Proof. The convergence in law to a Gaussian r.v. for $\varepsilon^{-1}(u^\varepsilon - u^0)$ follows after combining statements of Theorem 6 and Lemmata 5 (hence with the additional assumption **H1** when $\gamma = 1$, while for $\gamma > 1$ we don't need Lemma 5)and 6. Theorem 6 provides CLT for $\varepsilon^{-1}(\hat{u}^\varepsilon(x^j, t^j) - \check{u}^\varepsilon(x^j, t^j)))$, Lemma 5 ensures that $u^\varepsilon - \hat{u}^\varepsilon = o(\varepsilon^\gamma)$, Lemma 6 - that $\check{u}u^\varepsilon - u^0 = o(\varepsilon)$. $\square$

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